

Harmonic three-forms and sectional-curvature pinching in dimension six

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Abstract

We prove that a nonempty closed Riemannian six-manifold has vanishing third real cohomology and positive Euler characteristic if $K_{\min}(x) \geq (89/1000)K_{\max}(x) > 0$ at every point. The endpoint is included, with no orientation, symmetry, form-type, or global curvature-normalization assumption. The proof couples the Bochner formula to conservation of the complete quadratic stress pair and the two-form moment of a harmonic three-form, retaining the full first-order harmonic symbol. Exact rational certificates prove the finite inequalities. We retain the earlier 4/43 and 4000/43001 arguments to explain the successive differential completions. An L^3 curvature-deficit criterion also applies to positively curved conformal metrics on S^6 with pointwise pinching ratios tending to zero. These examples distinguish metric hypotheses, without claiming new topology. No optimal pinching threshold or unrestricted positive-sectional-curvature vanishing theorem is asserted.

Keywords. Harmonic forms; sectional curvature; conserved stress; pinching; Bochner formula; computer-assisted proof.

1 Introduction and main results

On a closed connected oriented six-manifold with positive sectional curvature, positive Ricci curvature gives $b_1 = b_5 = 0$, and duality gives

$$\chi(M) = 2 + 2b_2(M) - b_3(M).$$

Thus third-cohomology vanishing is a sufficient, stronger conclusion than Euler positivity. Sectional positivity alone does not make the ordinary three-form Bochner curvature endomorphism positive.

Our strongest explicit sufficient hypothesis is

$$\boxed{K_{\min}(x) \geq \frac{89}{1000}K_{\max}(x) > 0 \quad \forall x \quad \implies \quad H^3(M; \mathbb{R}) = 0, \quad \chi(M) > 0.} \quad (1.1)$$

The extrema are over two-planes at the same point. The theorem, proved in [theorem 6.1](#), applies to nonempty closed smooth six-manifolds, including nonorientable and disconnected ones. We first prove the 4/43 result in [theorem 4.1](#), since its exact equality type explains the mechanism. The strict symbol margin gives 4000/43001 in [theorem 5.1](#). Finally, the two-form moment supplies another integrated Weitzenböck identity. The complete curvature expression and its matching

harmonic-symbol estimate prove (1.1) in [theorem 6.1](#). None of these constants is claimed optimal for harmonic vanishing.

The second main result, [theorem 7.2](#), uses the potential

$$a_g = \frac{9}{26}(43K_{\min} - 4K_{\max})_+ - \frac{15}{26}(4K_{\max} - 43K_{\min})_+.$$

If $\|w\|_3^2 \leq S\|dw\|_2^2 + B\|w\|_2^2$, then $\|(a_0 - a_g)_+\|_3 < \min\{3/(4S), a_0/B\}$ for some $a_0 > 0$ forces $H^3(M; \mathbb{R}) = 0$. No curvature sign or separately assumed rough-Laplacian spectral gap is needed for that conclusion. [Proposition 7.4](#) constructs positively curved conformal metrics on S^6 with a prescribed ratio $r \in (0, 1]$ at a point, common Sobolev constants, and deficits tending to zero. This is an extension in metric hypotheses on known topology.

1.1 Relation to earlier work

The relation between pinching and Betti numbers was studied by Hulin [\[3\]](#). Seaman [\[6\]](#) proved $b_3 = 0$ under $1 \geq K > (4\sqrt{10} - 4)/(4\sqrt{10} + 23) = 0.2426\dots$. His proof establishes positivity of the three-form Weitzenböck curvature endomorphism at a fixed point; homogeneity gives the corresponding pointwise ratio version at the same threshold. Our comparison with that theorem is therefore the smaller sufficient ratio and the use of conserved stresses, not a claim that pointwise normalization itself is new.

There is also a concrete distinction in mechanism. The algebraic family in [lemma 4.4](#) has sectional range $[\delta, 1]$ and a unit three-form ρ for which

$$\langle q_3(R_\delta)\rho, \rho \rangle = (31\delta - 4)/3.$$

Consequently sectional pinching below $4/31 = 0.1290\dots$ cannot by itself force the ordinary three-form curvature term to be nonnegative. Our ratio $89/1000$ is below that barrier. The additional conserved-stress and moment identities overcome this particular pointwise obstruction; we do not assert a limitation on every Bochner-based global argument.

The symmetric-tensor curvature contractions fit the Weitzenböck framework of Bettiol–Mendes [\[1\]](#). Restricting first derivatives to the kernel of a natural differential symbol is also familiar from refined Kato inequalities [\[2\]](#); here the estimated quantity is the derivative of the entire quadratic stress pair. Petersen–Wink [\[5\]](#) prove $b_3 = 0$ on closed six-manifolds when the sum of the three lowest eigenvalues of the ordinary curvature operator on $\Lambda^2 TM$ is positive. Wink’s subsequent preprint [\[7, Theorem E\]](#) improves this condition to

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \frac{1}{2}\lambda_5 > 0$$

at every point, where $\lambda_1 \leq \dots \leq \lambda_{15}$; its nonnegative version makes all harmonic three-forms parallel. Nienhaus–Petersen–Wink [\[4, Theorem C\]](#) instead obtain $b_3 = 0$ under $36/5$ -positivity of the curvature operator of the second kind on trace-free symmetric two-tensors. These spectral hypotheses differ from the sectional-curvature ratio used here. We claim neither containment between the admitted metric classes nor an optimal threshold among all methods.

1.2 Proof and computational dependencies

Stress conservation and integration by parts give an exact global identity. Two finite inequalities supply its signs: one on the full harmonic first-jet kernel, the other on arbitrary algebraic curvature tensors. The geometric averaging and polytope enlargement are proved before enumerating

their vertices. Rational Gram identities and Schur pivots then verify the finite calculations. The earlier margin uses the same five original proof-package files unchanged, with a separate verifier. The moment completion supplies two further rational certificates and proof-only verifiers; the symbol verifier's exterior-algebra helpers are included explicitly. The supplement includes all these dependencies, an independent reduction check, provenance, and execution logs. The written global, covering, and conformal arguments remain part of the proof; test execution alone does not replace them.

2 Conventions, normal form, and curvature averaging

Let V be an oriented Euclidean vector space of dimension six. An algebraic curvature operator $R: \Lambda^2 V \rightarrow \Lambda^2 V$ is self-adjoint and satisfies the first Bianchi identity. We use the convention

$$R_{ij,kl} = \langle R^{\text{LC}}(e_i, e_j)e_l, e_k \rangle, \quad K_R(e_i, e_j) = R_{ij,ij}.$$

Thus the constant-sectional-curvature-one operator is I , and $\text{Scal}(I) = 30$. If ρ_{ij} is the infinitesimal orthogonal action on three-forms, the Weitzenböck curvature term is

$$q_3(R) = - \sum_{i < j, k < l} R_{ij,kl} \rho_{ij} \rho_{kl}, \quad q_3(I) = 9I. \quad (2.1)$$

In particular, $\langle q_3(R)\omega, \omega \rangle = \sum R_{ij,kl} \langle \rho_{ij}\omega, \rho_{kl}\omega \rangle$. Sectional estimates for such curvature terms are a classical ingredient of Bochner pinching arguments; see, for example, [6]. We give the finite-dimensional reductions needed for the precise constants used here.

2.1 Normal form and averaging

The operator $q_3(R)$ commutes with the Hodge star. Since $*^2 = -I$ on three-forms, its quadratic form is invariant under $\omega \mapsto \cos \theta \omega + \sin \theta * \omega$. After such a phase change and an oriented orthonormal change of frame, every real three-form has the normal form

$$\omega = x_0 e^{123} + x_1 e^{145} + x_2 e^{246} + x_3 e^{356}. \quad (2.2)$$

Here no ordering or sign restrictions on the x_i will be needed. For completeness, maximize the evaluation of a phase rotation of ω on an oriented orthonormal three-frame. Stationarity in the nine tangent directions to the Grassmannian removes the components of bidegree $(2, 1)$ relative to the maximizing plane and its orthogonal complement. Phase stationarity removes the complementary volume component. The remaining $(1, 2)$ component is a real 3×3 matrix; an oriented singular-value decomposition diagonalizes it, allowing signed diagonal entries. This gives (2.2).

Write W_0 for the four-dimensional space spanned by the forms in (2.2). Average R under the sixteen diagonal sign changes for which these four forms have the same sign character. This is a subgroup of $O(6)$; reflections are allowed, since each member sends ω to ω or $-\omega$. The average preserves sectional bounds and the value of the curvature quadratic form at ω . Its only surviving entries are the diagonal entries and the disjoint off-diagonal entries in the quartets

$$(2, 3, 4, 5), \quad (1, 3, 4, 6), \quad (1, 2, 5, 6).$$

Next average under the orthogonal transformation $Te_i = e_{7-i}$ for $i \leq 3$ and $Te_i = -e_{7-i}$ for $i \geq 4$. It normalizes the sign group and acts as $-*$ on W_0 , so this average also preserves the

required quadratic form. Denote the result by $\bar{R}$. These averages preserve scalar curvature as well.

For an ordered quartet (i, j, k, l) , introduce

$$\begin{aligned} a &= K_{ij} = K_{kl}, & b &= K_{ik} = K_{jl}, & c &= \frac{1}{2}(K_{il} + K_{jk}), \\ r &= \bar{R}_{ij,kl}, & s &= \bar{R}_{ik,jl}, & t &= \bar{R}_{il,jk}, & r - s + t &= 0, \\ u &= 2r - s, & v &= r - 2s. \end{aligned} \quad (2.3)$$

Suppose first that $0 \leq \sec_R \leq 1$. Coordinate planes and the six planes obtained from orthogonal sums and differences of pairs of the four coordinate vectors give the necessary inequalities

$$\begin{aligned} 0 &\leq a, b, c \leq 1, \\ 0 &\leq a + b \pm (u - v) \leq 2, \\ 0 &\leq a + c \pm u \leq 2, \quad 0 \leq b + c \pm v \leq 2. \end{aligned} \quad (2.4)$$

These inequalities are an outer relaxation of the sectional-curvature constraints; no sufficiency assertion is needed. For example, the unnormalized numerators on the planes $\langle e_i \pm e_k, e_j \pm e_l \rangle$ are $2(a + c \pm u)$, with the signs paired appropriately. Dividing by four gives the asserted interval. The other two pairs follow by permuting the four indices.

Lemma 2.1 (The local fourteen-vertex polytope). *The polytope (2.4) is the convex hull of*

$$(a, b, c, u, v) = (a, b, c, \varepsilon(a - c), \varepsilon(b - c)), \quad a, b, c \in \{0, 1\}, \quad \varepsilon \in \{-1, 1\}. \quad (2.5)$$

There are fourteen distinct points in this list.

Proof. Put $x = (a, b, c)$ and $d = (2u - v, -u + 2v, -u - v)/3$. If $A = x + d$ and $B = x - d$, the inequalities are exactly $0 \leq H_{ij} \leq 2$, where $H_{ij} = A_i + B_j$. The condition $\sum d_i = 0$ removes the redundant representation of this additive matrix. Write

$$H_{ij} = \tau + \alpha_i + \beta_j, \quad \tau = \min A + \min B, \quad \alpha_i = A_i - \min A, \quad \beta_j = B_j - \min B.$$

Then all three terms are nonnegative and $\tau + \max \alpha + \max \beta \leq 2$. Consequently $H/2$ is a convex combination of the all-ones matrix, a row-only matrix in the unit cube, a column-only matrix in the unit cube, and zero. Resolve the row and column vectors into binary cube vertices. Translating these matrices back gives (2.5); only the all-zero and all-one triples identify the two choices of sign. $\square$

For later quartic reconstruction, the matching tensors can be specified without a compression convention. Put $\ell_1 = (1, 6)$, $\ell_2 = (2, 5)$, $\ell_3 = (3, 4)$. For the h th quartet (i, j, k, l) , let $E_h(a, b, c, u, v)$ have diagonal entry $-c$ at ℓ_h , entry c at both other ℓ 's, entry a at (i, j) , (k, l) and entry b at (i, k) , (j, l) . Its only off-diagonal entries and symmetric partners are

$$(E_h)_{ij,kl} = \frac{2u - v}{3}, \quad (E_h)_{ik,jl} = \frac{u - 2v}{3}, \quad (E_h)_{il,jk} = -\frac{u + v}{3}. \quad (2.6)$$

The first Bianchi identity holds. These three tensors sum to the averaged curvature tensor: $c_1 = (K_{25} + K_{34})/2$, $c_2 = (K_{16} + K_{34})/2$, $c_3 = (K_{16} + K_{25})/2$, so for example $K_{16} = -c_1 + c_2 + c_3$. Individual summands and independent vertex choices need not satisfy the original sectional bounds. Only the necessary membership of each actual local parameter set in the polytope is used.

3 The complete conserved stress pair

3.1 The differential cost of a conserved symmetric tensor

For a real harmonic three-form ρ , write

$$F = |\rho|^2, \quad P = |T|^2 + |U|^2,$$

where the complete stress pair is

$$\begin{aligned} T(v, w) &= \langle \iota_v \rho, \iota_w \rho \rangle - \frac{1}{2} F g(v, w), \\ U(v, w) &= \frac{1}{2} (\langle \iota_v \rho, \iota_w * \rho \rangle + \langle \iota_w \rho, \iota_v * \rho \rangle). \end{aligned}$$

The complementary-contraction identity

$$\langle \iota_v * \rho, \iota_w * \rho \rangle = F g(v, w) - \langle \iota_v \rho, \iota_w \rho \rangle$$

follows from the Hodge identities for wedge and contraction, and gives $T_{*\rho} = -T_\rho$. For a constant Hodge phase $\rho_\theta = \cos \theta \rho + \sin \theta * \rho$, direct expansion gives

$$\begin{pmatrix} T_{\rho_\theta} \\ U_{\rho_\theta} \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} T_\rho \\ U_\rho \end{pmatrix}.$$

Both tensors are symmetric, trace-free, and divergence-free. The first conservation law is the differential-form stress identity, obtained by expanding the divergence and using $d\rho = \delta\rho = 0$. The second also follows from the first: $T_{*\rho} = -T_\rho$ and $T_{(\rho+*\rho)/\sqrt{2}} = U_\rho$, while the constant Hodge phase preserves harmonicity. No pointwise varying phase is used.

Explicitly, in a normal frame use antisymmetric components, with $F = \frac{1}{6} \sum_{i,a,b} \rho_{iab}^2$ and $T_{ij} = \frac{1}{2} \sum_{a,b} \rho_{iab} \rho_{jab} - \frac{1}{2} F \delta_{ij}$. Coclosedness says $\sum_i \nabla_i \rho_{iab} = 0$. Contracting

$$\nabla_i \rho_{jab} - \nabla_j \rho_{iab} + \nabla_a \rho_{ijb} - \nabla_b \rho_{ija} = 0$$

with ρ_{iab} and renaming indices gives $\sum_{i,a,b} \rho_{iab} \nabla_i \rho_{jab} = \nabla_j F$. Thus the two terms of $\text{div } T$ cancel. The trace identities are $\sum_i |\iota_{e_i} \rho|^2 = 3F$ and $\sum_i \langle \iota_{e_i} \rho, \iota_{e_i} * \rho \rangle = 3\langle \rho, * \rho \rangle = 0$. These computations are polynomial and hold at zeros. They do not assume that either stress is Codazzi.

For a symmetric tensor H , let

$$(C_H)_{ijk} = \nabla_i H_{jk} - \nabla_j H_{ik}, \quad Q_R(H) = \sum_{i < j} K_{ij} (\lambda_i - \lambda_j)^2,$$

where the latter expression is evaluated in an orthonormal eigenframe of H . It defines an invariant curvature contraction, independent of choices inside repeated eigenspaces. Norms of C_H sum over all three indices, without an additional factor of $1/2$.

Lemma 3.1 (Conserved-tensor coercivity). *On a closed n -manifold, every smooth symmetric divergence-free tensor satisfies*

$$\int |\nabla H|^2 \geq 2 \int Q_R(H).$$

If H is also trace-free and $\sec_x \geq m(x)$, then

$$\int |\nabla H|^2 \geq 2n \int m(x) |H|^2. \quad (3.1)$$

The tensor is not assumed to be Codazzi.

Proof. Integration by parts and the covariant-derivative commutator give

$$\frac{1}{2} \int |C_H|^2 + \int |\operatorname{div} H|^2 = \int |\nabla H|^2 + \int Q_R(H). \quad (3.2)$$

For clarity, expanding the leftmost norm gives $\int |\nabla H|^2 - \int \nabla_i H_{jk} \nabla_j H_{ik}$; commuting the two derivatives after integration by parts changes the second integral into $\int |\operatorname{div} H|^2 - \int Q_R(H)$. In an eigenframe the curvature contraction is the displayed sum of sectional curvatures times squared eigenvalue differences.

If $A_{ijk} = A_{ikj}$ and A^{sym} is its complete symmetrization, direct orthogonal decomposition gives

$$|A_{ijk} - A_{jik}|^2 = 3(|A|^2 - |A^{\operatorname{sym}}|^2) \leq 3|A|^2.$$

Apply this to $A = \nabla H$ in (3.2). When the divergence is zero, the first assertion follows. Finally

$$\sum_{i < j} (\lambda_i - \lambda_j)^2 = n \sum_i \lambda_i^2 - \left(\sum_i \lambda_i \right)^2$$

proves (3.1) for trace-free H . $\square$

Lemma 3.2 (Sharp derivative bound for the complete stress). *For any two real three-forms on an oriented Euclidean six-space,*

$$|dT_\rho[\eta]|^2 + |dU_\rho[\eta]|^2 \leq 6|\rho|^2|\eta|^2. \quad (3.3)$$

The coefficient six is optimal for this pointwise map estimate.

Proof. Write $\mathcal{S}(\rho) = (T_\rho, U_\rho)$. In the normal form

$$\rho = x_0 e^{123} + x_1 e^{145} + x_2 e^{246} + x_3 e^{356},$$

direct contraction gives

$$P = \frac{1}{2} F^2 + \sum_i x_i^4 + 12x_0 x_1 x_2 x_3, \quad (3.4)$$

$$\frac{3}{2} F^2 - P = 2((x_0 x_1 - x_2 x_3)^2 + (x_0 x_2 - x_1 x_3)^2 + (x_0 x_3 - x_1 x_2)^2). \quad (3.5)$$

The complete stress norm is invariant under oriented frame changes and constant Hodge phase, so $|\mathcal{S}(\rho)| \leq \sqrt{3/2}|\rho|^2$ holds for every form. Quadratic polarization, for $h > 0$, gives

$$|d\mathcal{S}_\rho[\eta]| = \frac{|\mathcal{S}(\rho + h\eta) - \mathcal{S}(\rho - h\eta)|}{2h} \leq \sqrt{\frac{3}{2}} \left(\frac{|\rho|^2}{h} + h|\eta|^2 \right).$$

Minimize at $h = |\rho|/|\eta|$; zero cases are immediate. For a unit simple form, equality holds with $\eta = \rho$ and also with $\eta = *\rho$. This proves sharpness. $\square$

Definition 3.3 (Factorized equality type). A nonzero form is self-dual-factorized if, for orthonormal covectors α, β and a unit self-dual two-form η on their orthogonal four-space W , oriented by $\operatorname{vol}_6 = \alpha \wedge \beta \wedge \operatorname{vol}_W$,

$$\rho = r\alpha \wedge \eta, \quad *\rho = r\beta \wedge \eta, \quad r > 0.$$

Indeed $*_6(\alpha \wedge \eta) = \beta \wedge *_W \eta$ with this convention. The choice of β , and hence the orientation of W , may change when the relative signs of the two normal-form coefficients change. In the four-coefficient normal form, two equal-magnitude nonzero coefficients have this type and satisfy $P = F^2 > 0$, by (3.4). This is a pointwise algebraic definition; no smooth choice of factors on a manifold is assumed.

Applying these lemmas to a harmonic form on a six-manifold yields

$$12 \int mP \leq \int (|\nabla T|^2 + |\nabla U|^2) \leq 6 \int F|\nabla \rho|^2. \quad (3.6)$$

This is the genuinely global input. Conservation, not a choice of form shape, makes the first inequality available.

4 Retaining the full harmonic symbol

Treating each derivative as an arbitrary three-form and discarding the completely symmetric stress derivative loses information. Retaining both the harmonic equations and this symmetric term gives the theorem below for every algebraic type of harmonic three-form.

Theorem 4.1 (Full-symbol pointwise pinching). *Let (M, g) be a nonempty closed Riemannian six-manifold. Suppose that at every point*

$$K_{\min}(x) \geq \frac{4}{43} K_{\max}(x) > 0.$$

Then $H^3(M; \mathbb{R}) = 0$ and $\chi(M) > 0$. Equality in the pinching ratio is included. No orientation, symmetry, form-shape, or global curvature normalization hypothesis is required.

The number $4/43$ is approximately 0.09302. In particular, pointwise 1/10-pinching suffices. We do not claim that this is the optimal geometric threshold. The proof below is complete under its pinching hypothesis; removing that hypothesis remains a separate problem.

4.1 The exact differential term

Use the stress pair $\mathcal{S} = (T, U)$ and the curvature contraction Q_R from Section 3. For a symmetric tensor H , set

$$(\text{Sym } \nabla H)_{ijk} = \frac{1}{3}(\nabla_i H_{jk} + \nabla_j H_{ki} + \nabla_k H_{ij}).$$

Define

$$\mathcal{J}_\rho(\nabla \rho) = |\nabla T|^2 + |\nabla U|^2 - 3|\text{Sym } \nabla T|^2 - 3|\text{Sym } \nabla U|^2.$$

The Codazzi identity (3.2) and the exact hook decomposition, rather than its inequality, give

$$2 \int_M (Q_R(T) + Q_R(U)) = \int_M \mathcal{J}_\rho(\nabla \rho). \quad (4.1)$$

Conservation of both stresses is essential here. For a general symmetric tensor the right side also contains the negative term $-2 \int |\text{div } H|^2$ relative to $2 \int Q_R(H)$.

Lemma 4.2 (Strict harmonic-symbol bound). *Let $\rho \in \Lambda^3(\mathbb{R}^6)^*$ and $\eta \in (\mathbb{R}^6)^* \otimes \Lambda^3(\mathbb{R}^6)^*$ satisfy*

$$\sum_i e^i \wedge \eta_i = 0, \quad \sum_i \iota_{e_i} \eta_i = 0.$$

Differentiate T, U algebraically using η_i in slot i , and form $\mathcal{J}_\rho(\eta)$ as above. Then

$$\mathcal{J}_\rho(\eta) \leq 3|\rho|^2|\eta|^2, \quad (4.2)$$

with strict inequality whenever both ρ and η are nonzero. The coefficient three is a certified bound, not asserted sharp.

Proof. All operations are equivariant under oriented orthogonal changes of frame and constant Hodge phase. Normalize ρ at the point to $x_0e^{123} + x_1e^{145} + x_2e^{246} + x_3e^{356}$. This frame and phase are held constant in the pointwise derivative calculation; no derivatives of a varying normal form are omitted.

Here are complete specifications of the rational certificate. Order the twenty coordinate triples lexicographically. Alternation and contraction form a 30×120 integer matrix K , with $KK^t = 4I_{30}$. Let N be the rational kernel basis obtained by row reduction, so $\eta = Ny$ with $y \in \mathbb{R}^{90}$, and put $G = N^tN$. If $A_{ijk}^T = dT_\rho[\eta_i]_{jk}$, and similarly for U , then

$$\mathcal{J}_\rho(\eta) = -2 \sum_{ijk} (A_{ijk}^T A_{jik}^T + A_{ijk}^U A_{jik}^U).$$

This identity reconstructs the quadratic matrix without a choice of symmetrization weights. Write that matrix in kernel coordinates as

$$H(x) = \sum_a x_a^2 H_{aa} + 2 \sum_{a < b} x_a x_b H_{ab}.$$

The common coordinate decomposition of G and the ten matrices H_{ab} has six blocks of size twelve and two of size nine. For each block B , the supplied rational skew matrices $Z_{ab,B}$ give a symmetric block Gram matrix

$$(\mathcal{G}_B)_{aa} = 3G_B - H_{aa,B}, \quad (\mathcal{G}_B)_{ab} = -H_{ab,B} + Z_{ab,B} \quad (a < b),$$

with the lower blocks fixed by transpose. Consequently

$$3|\rho|^2|\eta|^2 - \mathcal{J}_\rho(\eta) = \sum_B (x \otimes y_B)^t \mathcal{G}_B (x \otimes y_B). \quad (4.3)$$

The skew corrections contribute zero to this polynomial.

Every coefficient identity and every positive rational Schur-complement pivot is checked in `verify_symbol.py`. There are six 48×48 and two 36×36 positive-definite Gram matrices, with 360 positive pivots in total. The certificate uses only rational data; its verifier requires neither floating-point arithmetic nor an optimization solver. Thus every summand in (4.3) is nonnegative. If $x \neq 0$ and $\eta \neq 0$, some $y_B \neq 0$, and that summand is strictly positive. This proves both the estimate and its strictness, including all coefficient collisions. $\square$

Remark 4.3 (A limit of this differential estimate). The coefficient in (4.2) cannot be replaced by any number less than $8/3$, even if a nonnegative multiple of $|dF|^2$ is retained on the favorable side. Indeed, for $\rho = 3e^{123} + e^{145} + e^{246} + e^{356}$ take

$$\begin{aligned} \eta_1 &= -e^{256} + e^{346}, & \eta_2 &= e^{156} - e^{345}, & \eta_3 &= -e^{146} + e^{245}, \\ \eta_4 &= -e^{136} + e^{235}, & \eta_5 &= e^{126} - e^{234}, & \eta_6 &= -e^{125} + e^{134}. \end{aligned}$$

The two harmonic-symbol equations hold, $F = 12$, $|\eta|^2 = 12$, $dF = 0$, and $\mathcal{J} = 384$. These are exact finite data, not a global harmonic-form construction. They prevent decreasing this coefficient to zero as a way to remove curvature pinching. They establish a necessary lower bound, not a universal inequality at coefficient $8/3$ or sharpness of that endpoint.

4.2 The matching curvature estimate

Lemma 4.4 (Joint curvature at the new ratio). *For an algebraic curvature tensor with $4/43 \leq \sec_R \leq 1$, put $F = |\rho|^2$ and*

$$\mathcal{A}_R^{\text{harm}}(\rho) = Q_R(T) + Q_R(U) + \frac{3}{2}F\langle q_3(R)\rho, \rho \rangle.$$

Then $\mathcal{A}_R^{\text{harm}}(\rho) \geq 0$. If $\rho \neq 0$ and equality holds, ρ is self-dual-factorized. The ratio $4/43$ is sharp for this particular pointwise quartic.

Proof. The curvature averages and the fourteen-vertex enlargement in (2.4) preserve this contraction. The sixteen sign changes fix the normal form up to sign; orientation reversal merely reverses the cross-stress U . The additional orthogonal transformation acts as $-*$, sending (T, U) to $(-T, -U)$. Hence both averages preserve $Q_R(T) + Q_R(U)$ as well as $q_R(\rho)$, while preserving sectional bounds by convexity. Use the explicit matching reconstruction (2.6). Write $R = (4I + 39E)/43$. With $P = |T|^2 + |U|^2$ and $q_E = \langle q_3(E)\rho, \rho \rangle$, the integer quartic to certify is

$$86\mathcal{A}_R^{\text{harm}} = 48P + 108F^2 + 78(Q_E(T) + Q_E(U)) + 117Fq_E. \quad (4.4)$$

The 14^3 matching triples reduce, under the same coefficient permutations and even sign changes, to 84 quartics. The exact certificate in `verify_curvature.py` reconstructs all these contractions, the complete orbit coverage, every quartic coefficient, and positive semidefiniteness by rational elimination. There are 83 positive-definite Gram matrices and one rank-nine matrix. Its only nonzero real zeros are the factorized two-equal-coefficient forms. A nonzero quadratic monomial vector in its kernel has $x_2 = x_3 = 0$ and $x_0^2 = x_0x_1 = x_1^2$. Undoing coefficient permutations and even sign changes gives two equal-magnitude nonzero coefficients. A vanishing convex combination of nonnegative quartics requires every positive-weight summand to vanish; thus the same necessary equality condition holds for the original tensor. The two coordinate triples share one factor and define a self-dual two-form on the orthogonal four-space. In particular $P = F^2 > 0$ at every nonzero equality form; no smooth field of such factors is required.

For sharpness let J pair e_2, e_3 and e_4, e_5 , and vanish on the complementary two-plane. Define

$$\begin{aligned} E_J(X, Y, Z, W) &= \langle JX, Z \rangle \langle JY, W \rangle - \langle JX, W \rangle \langle JY, Z \rangle \\ &\quad + 2\langle JX, Y \rangle \langle JZ, W \rangle. \end{aligned}$$

This tensor satisfies Bianchi by expansion and has sectional numerator $3\langle JX, Y \rangle^2$. Thus $E_J/3$ has sectional range $[0, 1]$. For $R_\delta = \delta I + (1 - \delta)E_J/3$ and the unit form $\rho = (e^{123} + e^{145})/\sqrt{2}$, direct contraction gives

$$Q_{R_\delta}(T) + Q_{R_\delta}(U) = 6\delta, \quad \langle q_3(R_\delta)\rho, \rho \rangle = \frac{1}{3}(31\delta - 4),$$

hence $\mathcal{A}_{R_\delta}^{\text{harm}} = (43\delta - 4)/2$. This is an algebraic obstruction below the ratio, not a closed-metric counterexample or a claim of optimal topological pinching. $\square$

Remark 4.5 (An obstruction to ordinary pointwise Bochner positivity). The same algebraic family has strictly positive sectional curvature and a negative q_3 direction whenever $0 < \delta < 4/31$. Thus no implication from the sectional ratio alone to $q_3 \geq 0$ can hold throughout this interval. The pinching ratios proved here lie below this barrier: conservation and the integrated stress identities supply information absent from that pointwise sign test. This does not exclude other uses of the Bochner identity or impose a barrier on all global methods.

4.3 Global vanishing, including the endpoint

Proof of Theorem 4.1. Work first on an oriented connected component and suppose that $\rho \neq 0$ is harmonic. Put

$$E = \int F|\nabla\rho|^2, \quad D = \int |dF|^2, \quad C = \int (Q_R(T) + Q_R(U)), \quad \mathcal{G} = \int (3F|\nabla\rho|^2 - \mathcal{J}_\rho(\nabla\rho)).$$

The scalar Laplacian convention is $\Delta = \operatorname{div} \nabla$. The Bochner identity $\frac{1}{2}\Delta F = |\nabla\rho|^2 + q_R(\rho)$, multiplied by the smooth function F and integrated, and (4.1) give

$$E + \frac{1}{2}D + \int F\langle q_3(R)\rho, \rho \rangle = 0, \quad 2C = 3E - \mathcal{G}.$$

Apply Lemma 4.4 after dividing R_x by $K_{\max}(x)$, and restore the scale by linearity. No derivative of that scale enters. We obtain the exact global comparison

$$0 \leq \int \mathcal{A}_R^{\text{harm}}(\rho) = -\frac{1}{2}\mathcal{G} - \frac{3}{4}D \leq 0. \quad (4.5)$$

Thus F is a positive constant and $\mathcal{G} = 0$. Strictness in Lemma 4.2 forces $\nabla\rho = 0$, so $C = 0$ by (4.1). Meanwhile the nonnegative continuous integrand on the left of (4.5) vanishes everywhere. Lemma 4.4 makes ρ factorized, with $P = F^2 > 0$. Consequently

$$C \geq 6 \int K_{\min}(x)P \, d\operatorname{vol}_g > 0,$$

a contradiction. The constant norm eliminates all zero-set issues before this equality argument. For a general manifold apply the argument componentwise on its orientation cover. Pullback is injective on real cohomology. Positive Ricci curvature and duality give positive Euler characteristic on the cover, and its sign descends by multiplicativity under finite coverings. $\square$

4.4 What this changes, and what it does not

The improvement uses information absent from an arbitrary curvature tensor and form value: the coupled first derivatives of an actual harmonic form and a global integration-by-parts identity. The two finite certificates verify the pointwise estimates; the proof of (4.5) supplies their geometric implication. Neither calculation asserts that a formal harmonic jet is globally realizable.

The remaining unrestricted problem is not a missing endpoint argument. Even the new joint quartic is negative on strictly positive algebraic tensors below its threshold. A further proof must exploit information not captured by this fixed pointwise combination, or construct a different global mechanism. In particular, this theorem neither supplies the general twisting sign nor solves the universal zero-plane-measure problem.

There is also a quantitative limit to optimizing this exact proof template. Suppose one seeks constants $c, b \geq 0$ with

$$\mathcal{J}_\rho(\eta) + b|dF|^2 \leq cF|\eta|^2$$

for all harmonic first jets, and then proves pointwise nonnegativity of $Q_R(T) + Q_R(U) + (c/2)Fq_R$. Remark 4.3 forces $c \geq 8/3$. The factorized curvature witness forces

$$\delta \geq \frac{4c}{36 + 31c} \geq \frac{8}{89}.$$

Thus this fixed-coefficient, pointwise-completion template cannot reach arbitrary positive sectional curvature, even after optimizing its derivative constant or retaining a favorable norm-gradient term. This is not a barrier to other global harmonic identities, nonlinear weights, or a shape-dependent argument with controlled transition terms.

5 An explicit improvement from the strict certificate

The strict positive-definite symbol certificate has a quantitative margin. The following improvement uses the same rational data, without a new semidefinite optimization or a conjecture about the optimal coefficient.

Theorem 5.1 (Explicit improved pinching). *Let (M, g) be a nonempty closed Riemannian six-manifold. If at every point*

$$K_{\min}(x) \geq \frac{4000}{43001} K_{\max}(x) > 0,$$

then $H^3(M; \mathbb{R}) = 0$ and $\chi(M) > 0$. The endpoint is included. No orientation or connectedness hypothesis is required.

The ratio is $4000/43001 = 0.0930210925\dots$, strictly smaller than $4/43 = 0.0930232558\dots$. The gain is modest. Its purpose is to extract an explicit improvement already contained in the certificates and to separate sharpness of a fixed quartic from the strength of the method.

5.1 The reduced symbol coefficient

Put $c = 2999/1000$. In the notation of (4.3), replace each original Gram matrix by

$$\mathcal{G}_B^{(c)} = \mathcal{G}_B - (3 - c)(I_4 \otimes G_B).$$

The diagonal blocks become $cG_B - H_{aa,B}$; all mixed blocks and skew corrections stay unchanged. Therefore the exact coefficient identity is

$$c|\rho|^2|\eta|^2 - \mathcal{J}_\rho(\eta) = \sum_B (x \otimes y_B)^t \mathcal{G}_B^{(c)} (x \otimes y_B) > 0 \quad (5.1)$$

whenever $\rho \neq 0$ and the harmonic jet $\eta \neq 0$. The strict inequality follows from 360 strictly positive rational Schur pivots. The replay `verify_margin.py` reconstructs the full harmonic kernel and every coefficient matrix before checking these pivots. It reads the original `certificate.json` unchanged.

5.2 A strictly positive curvature completion

Write $F = |\rho|^2$, $P = |T|^2 + |U|^2$, $Q_R = Q_R(T) + Q_R(U)$, and

$$A_R^{(c)} = Q_R + \frac{c}{2} F q_R(\rho), \quad d = \frac{4000}{43001}, \quad d_0 = \frac{4}{43}, \quad t = \frac{d_0 - d}{1 - d_0} = \frac{4}{1677039}.$$

For any tensor with $d \leq \sec_R \leq 1$, the tensor $R_0 = (R + tI)/(1 + t)$ has $d_0 \leq \sec_{R_0} \leq 1$. Linearity and the trace-free tensor estimate give

$$\begin{aligned} A_R^{(c)} &= \frac{c}{3} A_R^{(3)} + (1 - \frac{c}{3}) Q_R \\ &\geq \frac{c(1+t)}{3} A_{R_0}^{(3)} + (6d(1 - c/3) - 2ct) P - \frac{9ct}{2} F^2. \end{aligned} \quad (5.2)$$

Here $A^{(3)}$ is exactly the quartic $\mathcal{A}^{\text{harm}}$ in lemma 4.4; $A_I^{(3)} = 6P + 27F^2/2$ and $Q_R \geq 6dP$. The last inequality is used only for the actual sectional-curvature tensor R , not for any relaxed vertex tensor.

The right side of (5.2) is strictly positive for every nonzero form, as the following explicit rational reconstruction proves. Let z be the ten quadratic monomials in lexicographic order. Write each quadratic entry of T, U as $v^t z$ and let $G_P = \sum v v^t$, the sum taken over all entries of both tensors. Thus $z^t G_P z = P$. Put $f = (1, 0, 0, 0, 1, 0, 0, 1, 0, 1)^t$, so $G_F = f f^t$ represents F^2 . For each of the 84 original curvature representatives $p = z^t G_p z$ of $86A_{R_0}^{(3)}$, define

$$M_p = \frac{c(1+t)}{258} G_p + \frac{5143}{29947125} G_P - \frac{9ct}{2} G_F. \quad (5.3)$$

The middle coefficient equals $6d(1 - c/3) - 2ct$ exactly. Every M_p has ten strictly positive rational Schur pivots. The replay reconstructs all original 2744 triples and 84 orbits, checks every coefficient of (5.3), and verifies these 840 pivots. No additional certificate file is needed.

Both P and F^2 are invariant under the coefficient permutations and even sign changes used in the original orbit reduction. Hence the same strict conclusion holds for every original vertex polynomial. Convex combinations preserve strict positivity at every nonzero x , since $z(x) \neq 0$. The geometric averaging preserves $A_{R_0}^{(3)}$. Consequently (5.2) proves

$$A_R^{(c)}(\rho) > 0 \quad \text{if } d \leq \sec_R \leq 1, \quad \rho \neq 0. \quad (5.4)$$

Proof of theorem 5.1. Work on an oriented connected component and let ρ be harmonic. The conserved-stress and weighted Bochner identities give, without any curvature sign assumption,

$$\int_M A_R^{(c)}(\rho) = -\frac{1}{2} \int_M (cF|\nabla \rho|^2 - \mathcal{J}_\rho(\nabla \rho)) - \frac{c}{4} \int_M |dF|^2 \leq 0$$

by (5.1). At each point apply (5.4) to $R_x/K_{\max}(x)$ and restore scale by linearity. The integrand on the left is nonnegative and strictly positive wherever $\rho \neq 0$. A nonzero smooth form is nonzero on an open set, so its integral would be positive, a contradiction. Thus $\rho = 0$. Transfer on the orientation cover, positive-Ricci vanishing of b_1 , and Poincaré duality give the full statement exactly as in theorem 4.1. $\square$

No optimality is claimed for c , for d , or for the choice of Gram representation. The integral criterion below retains the simpler $4/43$ potential and its stated constants. The explicit obstruction $8/89$ for the fixed-coefficient completion remains valid.

6 A stronger pinching theorem from the complete moment identity

Retaining the two-form moment in addition to the symmetric stress pair improves the pointwise pinching range. The moment is used with its full exterior derivative and codifferential; it is not assumed harmonic.

Theorem 6.1 (Moment-completed pinching). *Let (M, g) be a nonempty closed Riemannian six-manifold. If at every point*

$$K_{\min}(x) \geq \frac{89}{1000} K_{\max}(x) > 0,$$

then $H^3(M; \mathbb{R}) = 0$ and $\chi(M) > 0$. The endpoint is included. No orientation or connectedness hypothesis is required.

The ratio $89/1000 = 0.089$ improves $4000/43001$ in [theorem 5.1](#). It is also below the $8/89$ obstruction for the particular two-term completion in [Section 4](#). The additional moment term changes that completion. No optimal pinching constant or unrestricted positive-sectional-curvature conclusion is asserted.

6.1 The complete identity

On an oriented component, let ρ be a real harmonic three-form. Use the conserved symmetric tensors T, U of [Section 3](#), and put $F = |\rho|^2$. Define the real two-form

$$\mu_{ij} = \langle J_{ij}\rho, *\rho \rangle, \quad J_{ij}e^i = -e^j, \quad J_{ij}e^j = e^i.$$

Exterior norms count increasing index tuples. In particular,

$$|d^\nabla H|^2 = \sum_{i < j, k} (\nabla_i H_{jk} - \nabla_j H_{ik})^2$$

for a symmetric tensor H ; its ordinary tensor norm counts all ordered indices. Set

$$\begin{aligned} P_R &= Q_R(T) + Q_R(U), \\ D_S &= |d^\nabla T|^2 + |d^\nabla U|^2 - |\nabla T|^2 - |\nabla U|^2, \\ E &= F|\nabla \rho|^2 + \frac{1}{2}|dF|^2, \\ D_\mu &= |d\mu|^2 + |\delta\mu|^2 - |\nabla \mu|^2. \end{aligned}$$

Here $Q_R(H) = \sum_{i < j} K_{ij}(\lambda_i - \lambda_j)^2$ in an eigenframe of H . Fix $c = 1201/1000$ and define

$$\mathcal{C}_R(\rho) = P_R + cF\langle q_3(R)\rho, \rho \rangle - \frac{1}{20}\langle q_2(R)\mu, \mu \rangle. \quad (6.1)$$

The conserved-stress identity, weighted three-form Bochner identity, and ordinary two-form Weitzenböck identity give respectively

$$\int_M P_R = \int_M D_S, \quad \int_M F\langle q_3(R)\rho, \rho \rangle = - \int_M E, \quad \int_M \langle q_2(R)\mu, \mu \rangle = \int_M D_\mu.$$

Thus, without a curvature hypothesis,

$$\boxed{\int_M \mathcal{C}_R(\rho) = - \int_M \left(cE - D_S + \frac{1}{20}D_\mu \right)}. \quad (6.2)$$

Every coefficient is constant and every expression is polynomial in the smooth form and its derivatives. Zeros of ρ require no division, cutoff, regularity assumption, or omitted boundary term.

6.2 The full harmonic first-jet estimate

Lemma 6.2. *For a three-form value ρ and a first jet $\eta \in (\mathbb{R}^6)^* \otimes \Lambda^3(\mathbb{R}^6)^*$ satisfying*

$$\sum_i e^i \wedge \eta_i = 0, \quad \sum_i \iota_{e_i} \eta_i = 0,$$

differentiate T, U, μ, F using η in the preceding definitions. Then

$$cE - D_S + \frac{1}{20}D_\mu \geq 0, \quad (6.3)$$

strictly if $\rho \neq 0$ and $\eta \neq 0$.

Proof. Constant Hodge phase rotates the stress pair orthogonally and leaves μ unchanged. Indeed $*^2 = -I$, $*^* = -*$, and every infinitesimal rotation commutes with $*$. Expanding $\langle J(\cos \theta \rho + \sin \theta * \rho), *(\cos \theta \rho + \sin \theta * \rho) \rangle$ proves this directly. Apply the established oriented normal form to the value,

$$\rho = x_0 e^{123} + x_1 e^{145} + x_2 e^{246} + x_3 e^{356}.$$

The same constant frame and phase transform the entire jet; no moving normal-form choice is differentiated.

Here are full specifications of the exact certificate. In lexicographic exterior coordinates, the alternation and contraction matrix K has size 30×120 and satisfies $KK^t = 4I_{30}$. Eliminating its thirty distinct first pivot coordinates gives an integer matrix N with $KN = 0$ and ninety independent columns. Write $\eta = Ny$ and retain the induced norm $N^t N$.

The differentiated stresses, moment, and norm reconstruct all ten rational coefficient matrices L_{ij} in

$$cE - D_S + D_\mu/20 = \sum_i x_i^2 y^t L_{ii} y + 2 \sum_{i < j} x_i x_j y^t L_{ij} y.$$

They preserve six coordinate blocks of size twelve and two of size nine, covering all ninety harmonic directions. For each block B , the certificate supplies skew matrices $Z_{ij,B}$ with denominator 10^8 . Set

$$(\mathcal{G}_B)_{ii} = L_{ii,B}, \quad (\mathcal{G}_B)_{ij} = L_{ij,B} + Z_{ij,B} \quad (i < j),$$

and use transposes below the diagonal. The skew terms cancel in the exact polynomial identity

$$cE - D_S + D_\mu/20 = \sum_B (x \otimes y_B)^t \mathcal{G}_B (x \otimes y_B). \quad (6.4)$$

The proof-only replay `computation_artifacts/moment_pinching/verify_symbol.py` rebuilds every coefficient and checks every identity. Integer Bareiss elimination verifies all 360 positive leading principal minors: six matrices of size 48 and two of size 36. Sylvester's criterion therefore makes every $\mathcal{G}_B$ positive definite. If x and y are nonzero, at least one $x \otimes y_B$ is nonzero, proving strictness. The certificate supplies rational Gram entries, not numerical spectra or a solver verdict. $\square$

Remark 6.3 (A near-endpoint coefficient and its exact witness). For the form and harmonic jet of [remark 4.3](#), direct differentiation gives

$$E = 144, \quad D_S = 192, \quad |d\mu|^2 = 576, \quad |\delta\mu|^2 = 0, \quad |\nabla\mu|^2 = 192.$$

Thus $D_\mu = 384$ and, allowing the coefficient c to vary,

$$cE - D_S + D_\mu/20 = 144(c - 6/5).$$

Every coefficient valid for all harmonic jets must therefore satisfy $c \geq 6/5$, and strictness on nonzero pairs requires $c > 6/5$. Our certified 1201/1000 is 1/1000 above this necessary lower bound. The witness does not prove that 6/5 works for every jet; no sharp endpoint claim is needed for the theorem.

6.3 The matching curvature estimate

Lemma 6.4. *For every algebraic curvature tensor with $89/1000 \leq \sec_R \leq 1$ and every three-form ρ ,*

$$\mathcal{C}_R(\rho) \geq \frac{1}{1200}|\rho|^4. \quad (6.5)$$

Proof. We spell out why both curvature averages in Section 2 preserve all three contractions. Write $A \cdot R$ for the orthogonal action on curvature tensors. For $A \in O(6)$, the Hodge relation $*(A\rho) = \det(A)A(*\rho)$ and the tensor definitions give

$$T_{A\rho} = AT_\rho A^t, \quad U_{A\rho} = \det(A)AU_\rho A^t, \quad \mu(A\rho) = \det(A)A\mu(\rho).$$

Each stress and moment contraction is quadratic in its tensor, so these determinant signs disappear. Naturality of q_2, q_3 and Q_R therefore gives

$$\mathcal{C}_{A \cdot R}(A\rho) = \mathcal{C}_R(\rho). \quad (6.6)$$

Under a constant Hodge phase, (T, U) rotates orthogonally by twice the phase, preserving $Q_R(T) + Q_R(U)$. The norm and the q_3 energy are unchanged, since q_3 commutes with the isometric Hodge star. The phase calculation in the proof of lemma 6.2 gives $\mu(\rho_\theta) = \mu(\rho)$. Thus

$$\mathcal{C}_R(\cos \theta \rho + \sin \theta * \rho) = \mathcal{C}_R(\rho). \quad (6.7)$$

Every member of the sixteen-element sign group sends a normal-form ρ to $\pm \rho$, so (6.6) preserves its full curvature functional. The second signed reversal, denoted T in Section 2, has determinant $+1$ and acts as $-*$ on the normal-form space. Its inverse sends ρ to $*\rho$. Equations (6.6) and (6.7) therefore preserve the same functional under the second average. Finally all three terms are linear in R , so group averaging preserves their sum exactly. No varying phase or moving frame is differentiated in this algebraic reduction.

In the preceding normal form the moment is exactly

$$\mu = 2(-x_0x_1 + x_2x_3)e^{16} + 2(x_0x_2 - x_1x_3)e^{25} + 2(-x_0x_3 + x_1x_2)e^{34}.$$

Write $R = (89I + 911E)/1000$. The averaged E is the sum of the three full matching tensors (2.6). Each actual five-tuple lies in its fourteen-vertex polytope. Their product convex combination therefore reduces the assertion to all $14^3 = 2744$ reconstructed triples. This uses the full tensor, including the distinguished negative diagonal contributions in each summand, rather than just its compressed q_3 matrix. Individual vertex tensors need not satisfy the sectional bounds.

Let $z = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2)^t$. The replay `computation_artifacts/moment_pinching/verify_curvature.py` reconstructs all three curvature contractions from exterior rotations. After multiplication by 3,000,000, all 2744 quartics have integer coefficients. Permutations of the four coefficients and even sign changes, modulo common sign, give 96 polynomial actions. Computing the actions on the new complete quartics gives 84 orbits. For each representative p , rational free Gram entries specify G_p such that

$$p = z^t G_p z, \quad G_p - 5000I \succ 0.$$

The replay verifies all 35 quartic coefficients per representative, complete orbit coverage, and 840 strictly positive rational Schur pivots for these shifted matrices. No optimization output enters

the verification. Permutations and signs preserve both $|x|$ and $|z|$, so the same bound holds on every orbit and its convex combinations. Finally

$$|z|^2 = \sum_i x_i^4 + \sum_{i < j} x_i^2 x_j^2 \geq \frac{1}{2} \left(\sum_i x_i^2 \right)^2.$$

Dividing the certified Gram bound by 3,000,000 gives (6.5). $\square$

6.4 Global consequence and scope

Proof of theorem 6.1. For a harmonic ρ on an oriented connected component, lemma 6.2 and (6.2) imply $\int \mathcal{C}_R(\rho) \leq 0$. At each point normalize the actual curvature tensor by $K_{\max}(x)$ and apply lemma 6.4. Restoring scale gives

$$0 \geq \int_M \mathcal{C}_R(\rho) \geq \frac{1}{1200} \int_M K_{\max}(x) |\rho|^4 \geq 0. \quad (6.8)$$

This is a pointwise evaluation, not a conformal metric change; no derivative of $K_{\max}$ occurs. Positivity of $K_{\max}$ forces $\rho = 0$. Thus $b_3 = 0$. Positive Ricci curvature gives $b_1 = 0$, and Poincaré duality gives $\chi = 2 + 2b_2 > 0$ on each component. For a nonorientable component apply the argument to its orientation cover. Pullback is injective on real cohomology and Euler characteristic multiplies under finite covers, giving the stated result. $\square$

There is still a positive threshold in this exact completion. At $w = e^{123} + e^{145}$ and $R_\delta = \delta I + (1 - \delta)E_J/3$, with J pairing 23 and 45, the three round contractions are 24, 36, 32, while their E_J contractions are 0, -16 , 0. Consequently

$$\mathcal{C}_{R_\delta}(w) = \frac{54031\delta - 4804}{750}.$$

It is positive at $\delta = 89/1000$, with value $4759/750000$, and negative below $4804/54031$. If c varies while the moment weight remains $-1/20$, the same calculation gives

$$\mathcal{C}_{R_\delta, c}(w) = \frac{(336 + 620c)\delta - 80c}{15}.$$

The required lower ratio is therefore at least $80c/(336 + 620c)$, an increasing function of $c > 0$. Together with remark 6.3, this leaves the necessary barrier $4/45 = 0.0888\dots$ even when c is adjusted. These barriers concern this fixed moment weight and pointwise nonnegative completion; they establish neither attainability nor an optimal topological threshold. The round product $S^3 \times S^3$ retains its parallel factor-volume forms and zero complete curvature cost; it does not satisfy the positive pinching hypothesis. The weak nonnegative Euler assertion remains open.

7 Full-symbol integral curvature deficit

The full harmonic-symbol estimate in Section 4 also yields an integral curvature criterion. Its potential is an explicit function of the two sectional extrema, and its proof requires no separate positive rough-Laplacian gap. This section proves the criterion and constructs positively curved metrics satisfying it with arbitrarily small positive pointwise curvature ratios. The construction changes the metric hypothesis on the sphere; it supplies no new topology.

7.1 The scalar potential retained by the full symbol

All forms used in the stress argument below are real. On an oriented Riemannian six-manifold, for a smooth three-form ω put

$$\begin{aligned} F &= |\omega|^2, \\ T_{ij} &= \langle \iota_{e_i} \omega, \iota_{e_j} \omega \rangle - \frac{1}{2} F g_{ij}, \\ U_{ij} &= \frac{1}{2} (\langle \iota_{e_i} \omega, \iota_{e_j} * \omega \rangle + \langle \iota_{e_j} \omega, \iota_{e_i} * \omega \rangle). \end{aligned}$$

These are the stresses of Section 3. For a symmetric tensor H , the invariant contraction $Q_R(H)$ is $\sum_{i < j} K_{ij}(\lambda_i - \lambda_j)^2$ in an orthonormal eigenframe. Use the complete symmetrization convention

$$(\text{Sym } \nabla H)_{ijk} = \frac{1}{3} (\nabla_i H_{jk} + \nabla_j H_{ki} + \nabla_k H_{ij}),$$

and write

$$\begin{aligned} \mathcal{J}_\omega &= |\nabla T|^2 + |\nabla U|^2 - 3 |\text{Sym } \nabla T|^2 - 3 |\text{Sym } \nabla U|^2, \\ \mathcal{A}_R^{\text{harm}}(\omega) &= Q_R(T) + Q_R(U) + \frac{3}{2} F \langle q_3(R) \omega, \omega \rangle. \end{aligned} \tag{7.1}$$

In particular, this is the full-symbol quartic of Lemma 4.4, with coefficient 3/2 in its last term.

We use three established pointwise inputs:

$$\begin{aligned} |T|^2 + |U|^2 &\leq \frac{3}{2} F^2, \\ \mathcal{J}_\omega &\leq 3F |\nabla \omega|^2 \quad \text{if } d\omega = \delta\omega = 0, \\ \mathcal{A}_R^{\text{harm}}(\omega) &\geq 0 \quad \text{if } K_{\min}(R) \geq \frac{4}{43} K_{\max}(R) > 0. \end{aligned} \tag{7.2}$$

The first is (3.5); the other two are Lemmas 4.2 and 4.4, respectively. Their exact rational certificates and normalization conventions remain in that canonical proof. Scaling the curvature tensor by its positive maximum gives the last formulation in (7.2); no derivative of the scale is taken.

For an ordinary harmonic ω on a closed manifold, the equality underlying (4.5) is valid without a curvature sign:

$$\int_M \mathcal{A}_R^{\text{harm}}(\omega) d\text{vol}_g = -\frac{1}{2} \int_M (3F |\nabla \omega|^2 - \mathcal{J}_\omega) d\text{vol}_g - \frac{3}{4} \int_M |dF|^2 d\text{vol}_g. \tag{7.3}$$

Indeed, the weighted Bochner identity is

$$\int_M F |\nabla \omega|^2 d\text{vol}_g + \frac{1}{2} \int_M |dF|^2 d\text{vol}_g + \int_M F \langle q_3(R) \omega, \omega \rangle d\text{vol}_g = 0,$$

while stress conservation gives $2 \int_M (Q_R(T) + Q_R(U)) d\text{vol}_g = \int_M \mathcal{J}_\omega d\text{vol}_g$ by (4.1). Substitution proves (7.3), including its coefficient 3/4.

Lemma 7.1 (Explicit lower potential for the full-symbol quartic). *For the sectional extrema of an algebraic curvature tensor on an oriented Euclidean six-space, set*

$$\begin{aligned} \sigma_R &= 43K_{\min}(R) - 4K_{\max}(R), \\ a_R &= \frac{9}{26}(\sigma_R)_+ - \frac{15}{26}(-\sigma_R)_+, \quad s_+ = \max\{s, 0\}. \end{aligned} \tag{7.4}$$

For every real three-form ω and every such tensor R ,

$$\mathcal{A}_R^{\text{harm}}(\omega) \geq a_R |\omega|^4. \quad (7.5)$$

Consequently, with $\sigma_g(x) = \sigma_{R_x}$ and $a_g(x) = a_{R_x}$, every ordinary harmonic real three-form on a closed oriented six-manifold satisfies

$$\frac{3}{4} \|dF\|_2^2 + \int_M a_g F^2 d\text{vol}_g \leq 0, \quad F = |\omega|^2. \quad (7.6)$$

Proof. Let $l = K_{\min}(R)$, $u = K_{\max}(R)$, and let I denote the constant-curvature-one tensor. Put $\mu = (43l - 4u)/39$. If $u > l$, then $\bar{R} = R - \mu I$ has extrema

$$K_{\min}(\bar{R}) = \frac{4}{39}(u - l), \quad K_{\max}(\bar{R}) = \frac{43}{39}(u - l).$$

Thus $\bar{R}$ satisfies the last hypothesis in (7.2) exactly, whether or not the original R has positive sectional curvature.

Both T and U are trace-free for any real ω : contraction over an orthonormal frame gives $\text{tr } T = 3F - 3F = 0$ and $\text{tr } U = 3\langle \omega, *\omega \rangle = 0$, since $*^2 = -I$ and $*^* = -*$ on real three-forms in dimension six. For a trace-free symmetric tensor H ,

$$Q_I(H) = \sum_{i < j} (\lambda_i - \lambda_j)^2 = 6|H|^2, \quad q_3(I) = 9I.$$

Writing $P = |T|^2 + |U|^2$, we obtain

$$\mathcal{A}_I^{\text{harm}} = 6P + \frac{27}{2}F^2, \quad \frac{27}{2}F^2 \leq \mathcal{A}_I^{\text{harm}} \leq \frac{45}{2}F^2. \quad (7.7)$$

The upper bound uses the first inequality in (7.2). Linearity in curvature now gives

$$\mathcal{A}_R^{\text{harm}} = \mathcal{A}_{\bar{R}}^{\text{harm}} + \mu \mathcal{A}_I^{\text{harm}} \geq \begin{cases} \frac{27}{2} \mu F^2, & \mu \geq 0, \\ \frac{45}{2} \mu F^2, & \mu < 0. \end{cases}$$

Since $(27/2)/39 = 9/26$ and $(45/2)/39 = 15/26$, this is (7.5). If $u = l$, sectional-curvature polarization gives $R = lI$. In this case $\mu = l$, and the same two bounds follow directly from (7.7), including $l \leq 0$.

For harmonic ω , combine (7.5), (7.3), and the harmonic-symbol inequality in (7.2). This proves (7.6). $\square$

The sectional extrema, and hence a_g , are continuous: they are the extrema of the continuous sectional-curvature function over the compact two-plane Grassmannian in each tangent space. They need not be smooth, and no derivative of a_g is used. The function $F = |\omega|^2$ is smooth even at zeros of ω . Every identity above extends there directly; there is no division by F or excision of a nodal set.

7.2 An integral vanishing criterion

All L^p norms and volumes in this section use ordinary Riemannian volume, without volume normalization. For a scalar function, write $\|w\|_p = (\int_M |w|^p d\text{vol}_g)^{1/p}$.

Theorem 7.2 (Full-symbol L^3 -deficit vanishing). *Let (M^6, g) be closed, connected, and oriented. Suppose that $S, B > 0$ are constants in the scalar Sobolev inequality*

$$\|w\|_3^2 \leq S\|dw\|_2^2 + B\|w\|_2^2, \quad w \in W^{1,2}(M; \mathbb{R}). \quad (7.8)$$

For $a_0 > 0$ set $V_{a_0} = (a_0 - a_g)_+$. If

$$\|V_{a_0}\|_3 < \min \left\{ \frac{3}{4S}, \frac{a_0}{B} \right\}, \quad (7.9)$$

then $H^3(M; \mathbb{R}) = 0$. If additionally $\sec_g > 0$, then $\chi(M) = 2 + 2b_2(M) > 0$.

Proof. Let ω be an ordinary harmonic real three-form and put $F = |\omega|^2$. Since $a_g \geq a_0 - V_{a_0}$, (7.6) and Hölder's inequality give

$$\frac{3}{4}\|dF\|_2^2 + a_0\|F\|_2^2 \leq \int_M V_{a_0} F^2 d\text{vol}_g \leq \|V_{a_0}\|_3 \|F^2\|_{3/2} = \|V_{a_0}\|_3 \|F\|_3^2.$$

Apply (7.8) to the smooth scalar F . The result is

$$\left(\frac{3}{4} - S\|V_{a_0}\|_3 \right) \|dF\|_2^2 + (a_0 - B\|V_{a_0}\|_3) \|F\|_2^2 \leq 0.$$

Both coefficients are strictly positive under (7.9), so $F = 0$ and $\omega = 0$. Hodge theory proves the cohomology conclusion. If $\sec_g > 0$, positive Ricci curvature gives $b_1 = 0$; Poincaré duality gives $b_5 = b_1$, $b_4 = b_2$, and $b_6 = b_0 = 1$. Together with $b_3 = 0$, these equalities prove the stated Euler formula. $\square$

The argument is for ordinary real harmonic three-forms. Complex cohomology vanishes as well, because the real and imaginary parts of an ordinary harmonic complex form are real harmonic forms. No Hermitian-complex extension of the stress quartic is being assumed.

Corollary 7.3 (A deficit-volume bound). *Let (M^6, g) and the scalar Sobolev constants S, B be as in Theorem 7.2. Suppose $a_0 > 0$, $A \geq 0$, and a measurable set $U \subset M$ satisfy $a_g \geq a_0$ almost everywhere on $M \setminus U$ and $a_g \geq -A$ almost everywhere on U . Then*

$$\|V_{a_0}\|_3 \leq (a_0 + A) \text{Vol}_g(U)^{1/3}. \quad (7.10)$$

In particular, the vanishing conclusion holds if the right side of (7.10) is strictly less than the threshold in (7.9).

Proof. The deficit is zero off U and at most $a_0 + A$ on U . Cubing and integrating gives (7.10). $\square$

7.3 Localized positive metrics with arbitrarily small ratios

Proposition 7.4 (Arbitrary positive pointwise ratios within the criterion). *Let g_0 be the unit-round metric on S^6 and fix $p \in S^6$. There are constants $S_*, B_*, C > 0$, independent of $r \in (0, 1]$, with the following property. For each such r and every sufficiently small $\varepsilon > 0$ depending on r , there is a smooth conformal metric $g_{r,\varepsilon} = e^{2f_{r,\varepsilon}} g_0$ satisfying*

$$\begin{aligned} \sec_{g_{r,\varepsilon}} &> 0, & \frac{K_{\min}(p, g_{r,\varepsilon})}{K_{\max}(p, g_{r,\varepsilon})} &= r, \\ \|(1 - a_{g_{r,\varepsilon}})_+\|_{L^3(g_{r,\varepsilon})} &\leq C\varepsilon^{7/6}. \end{aligned} \quad (7.11)$$

The scalar Sobolev inequality (7.8) holds with the common constants S_*, B_* . For all sufficiently small such ε , these metrics satisfy Theorem 7.2 with $a_0 = 1$. Their sectional curvatures have a common finite upper bound, their conformal perturbations are supported on sets of volume $O(\varepsilon^{7/2})$, and $g_{r,\varepsilon} \rightarrow g_0$ in C^1 as $\varepsilon \rightarrow 0$, with these estimates uniform in r .

Proof. First choose the cutoffs independently of r . There is a fixed smooth compactly supported function $h : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$h(0) = -1, \quad h'(0) = 0, \quad h''(0) = 1, \quad h'' \leq 1. \quad (7.12)$$

Here is a construction retaining the last bound. Let a smooth cutoff χ_R take values in $[0, 1]$, equal one on $[-R, R]$, and vanish outside $[-R-1, R+1]$, with first and second derivative bounds independent of $R \geq 1$. Such cutoffs come from translating one fixed transition function on each end. Put $h(t) = -\chi_R(t)e^{-t^2/2}$. On the central interval, $h''(t) = (1-t^2)e^{-t^2/2} \leq 1$. On the transition intervals, the product rule bounds $|h''|$ by a fixed constant times $(1+|t|+t^2)e^{-t^2/2}$, which tends uniformly to zero as $R \rightarrow \infty$. Fix a sufficiently large R once and for all. This proves (7.12); the lower bound of h'' is also finite. Fix a smooth compactly supported $\theta : \mathbb{R}^5 \rightarrow [0, 1]$ equal to one on a neighborhood of zero.

In g_0 -normal coordinates $(x_1, x') \in \mathbb{R} \times \mathbb{R}^5$ centered at p , set $c = 1 - r \in [0, 1)$ and define

$$f_{r,\varepsilon}(x) = c\varepsilon^2 h(x_1/\varepsilon)\theta(x'/\sqrt{\varepsilon}), \quad g_{r,\varepsilon} = e^{2f_{r,\varepsilon}}g_0, \quad (7.13)$$

extending $f_{r,\varepsilon}$ by zero. A common upper bound on ε ensures that its support is compactly contained in the coordinate chart, so the extension is smooth. The support has length $O(\varepsilon)$ in the first coordinate and radius $O(\sqrt{\varepsilon})$ in the remaining five. Its g_0 -volume is therefore $O(\varepsilon^{7/2})$. All constants below depend only on the fixed cutoffs and chart, since $0 \leq c \leq 1$.

Direct differentiation yields

$$\begin{aligned} \|f_{r,\varepsilon}\|_\infty &\leq C_0\varepsilon^2, & \|df_{r,\varepsilon}\|_\infty &\leq C_1\varepsilon, \\ \nabla_0^2 f_{r,\varepsilon} &= ch''(x_1/\varepsilon)\theta(x'/\sqrt{\varepsilon})dx_1 \otimes dx_1 + O(\sqrt{\varepsilon}). \end{aligned} \quad (7.14)$$

The remainder is in g_0 -operator norm, uniformly on the sphere. Indeed, mixed coordinate second derivatives are $O(\sqrt{\varepsilon})$ and purely transverse ones are $O(\varepsilon)$. On the support $|x| = O(\sqrt{\varepsilon})$; the normal-coordinate metric error is $O(\varepsilon)$ and the connection error in the Hessian is $O(|x||df|) = O(\varepsilon^{3/2})$.

For any g_0 -orthonormal two-plane P , the trace of the rank-one leading Hessian in (7.14) is at most $c + O(\varepsilon)$. If its coefficient is negative the trace is nonpositive; otherwise use $h''\theta \leq 1$ and $|dx_1|_{g_0}^2 = 1 + O(\varepsilon)$. The conformal sectional-curvature formula therefore gives a constant C_2 independent of r such that

$$\begin{aligned} K_{g_{r,\varepsilon}}(P) &= e^{-2f_{r,\varepsilon}}(1 - \text{tr}_P \nabla_0^2 f_{r,\varepsilon} + |\text{pr}_P \nabla_0 f_{r,\varepsilon}|^2 - |\nabla_0 f_{r,\varepsilon}|^2) \\ &\geq e^{-2f_{r,\varepsilon}}(r - C_2\sqrt{\varepsilon}). \end{aligned} \quad (7.15)$$

For a fixed $r > 0$, choose $C_2\sqrt{\varepsilon} < r/2$ to obtain strictly positive sectional curvature. The uniformly bounded full Hessian, gradient, and conformal factor also give a common finite upper sectional bound.

At p , the gradient is zero and the Hessian is exactly $c dx_1 \otimes dx_1$. The possible two-plane traces of this Hessian range from zero to c . Thus

$$K_{\min}(p, g_{r,\varepsilon}) = e^{-2f_{r,\varepsilon}(p)}r, \quad K_{\max}(p, g_{r,\varepsilon}) = e^{-2f_{r,\varepsilon}(p)}c.$$

This proves the exact ratio in (7.11). When $r = 1$, $c = 0$ and the metric is round.

Outside the support of the perturbation, both sectional extrema are one and hence $a_{g_{r,\varepsilon}} = 27/2$. The deficit $V_1 = (1 - a_{g_{r,\varepsilon}})_+$ is supported inside that set. If M_* is the common upper sectional bound, positivity gives $\sigma_{g_{r,\varepsilon}} \geq -4M_*$, so (7.4) implies $0 \leq V_1 \leq 1 + (30/13)M_*$. Its supremum is therefore bounded independently of r and ε . The conformal volume element is $e^{6f_{r,\varepsilon}} d\text{vol}_{g_0}$, so

$$\|V_1\|_{L^3(g_{r,\varepsilon})} \leq C_3 \text{Vol}_{g_{r,\varepsilon}}(\text{supp } f_{r,\varepsilon})^{1/3} \leq C\varepsilon^{7/6}. \quad (7.16)$$

Finally, take any scalar Sobolev constants $S_0, B_0 > 0$ for g_0 . For $g = e^{2f}g_0$ with $\|f\|_\infty \leq \delta$, the scalar norm and Dirichlet-energy transformations in dimension six give

$$\|w\|_{3,g}^2 \leq e^{4\delta} \|w\|_{3,g_0}^2, \quad \|dw\|_{2,g_0}^2 \leq e^{4\delta} \|dw\|_{2,g}^2, \quad \|w\|_{2,g_0}^2 \leq e^{6\delta} \|w\|_{2,g}^2.$$

It follows that $S_g = e^{8\delta} S_0$ and $B_g = e^{10\delta} B_0$ are valid Sobolev constants for g . Shrink the common upper bound on ε until $C_0\varepsilon^2 \leq (\log 2)/10$. Then $S_* = 2S_0$ and $B_* = 2B_0$ work for all r . Taking ε still smaller until

$$C\varepsilon^{7/6} < \min\{3/(4S_*), 1/B_*\}$$

proves (7.9) with $a_0 = 1$. The first two estimates in (7.14) imply the uniform C^1 convergence of the metrics. $\square$

The positivity margin in (7.15) is essential: no common support scale is asserted to work as $r \downarrow 0$. For any sequence $r_j \downarrow 0$, choose $\varepsilon_j \downarrow 0$ with $\sqrt{\varepsilon_j}/r_j \rightarrow 0$ and with the integral threshold above. The corresponding metrics have common scalar Sobolev constants, their deficits tend to zero, and

$$\inf_{x \in S^6} \frac{K_{\min}(x, g_{r_j, \varepsilon_j})}{K_{\max}(x, g_{r_j, \varepsilon_j})} \leq r_j \rightarrow 0.$$

Thus the class admitted by Theorem 7.2 has no positive universal pointwise-ratio lower bound. There is no assertion that the minimum ratio is attained at p . These metrics are all conformal to the round sphere, whose middle cohomology already vanishes.

7.4 The integral criterion and the pointwise endpoint

The relation to the endpoint theorem is also precise. If $a_g \equiv 0$, then $V_{a_0} \equiv a_0$ and $\|V_{a_0}\|_3 = a_0 \text{Vol}_g(M)^{1/3}$. Testing (7.8) on $w = 1$ gives $B \geq \text{Vol}_g(M)^{-1/3}$, so the strict condition (7.9) cannot hold for any $a_0 > 0$. Theorem 4.1, however, includes the pointwise 4/43 boundary by its separate equality argument. The integral criterion is therefore not a replacement for that endpoint proof. Conversely, Proposition 7.4 admits positive metrics with ratios below every fixed positive pinching threshold. These statements give the respective strengths without identifying the two hypotheses.

These conservation and symbol identities concern ordinary real harmonic forms. Fields solving inhomogeneous equations require their source and divergence terms to be retained. No unrestricted positive-curvature conclusion follows.

A Exact certificate reconstruction and reproducibility

This section specifies the finite data for the harmonic-symbol pinching theorems. The self-contained supplement preserves the five original files and their relative paths: two verifiers,

two rational data files, and their exterior-algebra dependency. The margin verifier reconstructs its new matrices from those unchanged inputs. No numerical eigensolver, optimization solver, expected-output report, or internet access is used by the mathematical checks. Python and SymPy are required.

A.1 The full harmonic kernel and symbol Grams

Order increasing triples lexicographically, and jet coordinates first by derivative index and then triple. For coordinate (i, I) , its alternation coefficient is the sorting sign of (i, I) if $i \notin I$. If i occupies position j of I , starting at zero, its contraction coefficient is $(-1)^j$ in $I \setminus \{i\}$. Every coordinate occurs in exactly one of the thirty rows; every row has four nonzero entries. Hence $KK^t = 4I_{30}$. In each row eliminate its least-indexed nonzero column p . For a free column j in that row, the corresponding column of N is $e_j - (K_{ij}/K_{ip})e_p$. Thus $KN = 0$, and the free-row submatrix of N is I_{90} , proving that the whole kernel is represented.

Let B_ρ have rows $\iota_{e_i}\rho$, and let $C_\rho = B_{*\rho}$. The polynomial formulas

$$T = B_\rho B_\rho^t - \frac{1}{2}FI_6, \quad U = \frac{1}{2}(B_\rho C_\rho^t + C_\rho B_\rho^t)$$

define their derivatives without any normalizing-frame derivative. The checker reconstructs both the complete symmetrization expression and $-2\sum_{ijk}(A_{ijk}^T A_{jik}^T + A_{ijk}^U A_{jik}^U)$, and verifies their equality for all ten normal-form coefficients.

The common blocks, in zero-based kernel coordinates, are

Block	Coordinates
0	0, 12, 16, 23, 30, 37, 44, 48, 65, 67, 79, 86
1	1, 8, 13, 15, 45, 47, 50, 57, 64, 68, 78, 83
2	2, 7, 14, 21, 39, 46, 58, 63, 70, 77, 84, 88
3	3, 10, 26, 28, 32, 34, 51, 55, 62, 69, 76, 81
4	4, 9, 20, 27, 33, 40, 56, 61, 71, 75, 82, 89
5	5, 19, 24, 36, 41, 53, 60, 72, 74
6	6, 17, 22, 38, 43, 59, 66, 85, 87
7	11, 18, 25, 29, 31, 35, 42, 49, 52, 54, 73, 80

Their disjoint coverage and all off-block zeros are checked. Equivalently, form the graph whose vertices are the ninety kernel coordinates and join two coordinates when an off-diagonal entry of G or one of the ten coefficient matrices is nonzero. The displayed sets are its connected components. The moment coefficient matrices use the same decomposition; their ten off-block-zero identities are checked separately. No symmetry assumption on the jet is imposed. Each record (a, b, i, j, z) with $a < b$ and $i < j$ defines skew entries $Z_{ab,ij} = z$, $Z_{ab,ji} = -z$; unlisted entries are zero. The diagonal and mixed identities in (4.3) are then checked exactly. The original and reduced-coefficient matrices each have 360 positive rational pivots.

A.2 Curvature orbit and Gram reconstruction

Sort the fourteen vertices lexicographically and use (2.6). The matrices L_{ij} have entries $(L_{ij})_{ij} = 1$, $(L_{ij})_{ji} = -1$; D_{ij} denotes their induced action on three-forms. The contractions are

$$q_R = \sum_{a,b} R_{ab} \langle D_a \rho, D_b \rho \rangle, \quad Q_R(H) = \frac{1}{2} \sum_{a,b} R_{ab} \operatorname{tr}([L_a, H][L_b, H]).$$

There is no extra factor of one half in q_R . Represent quartics by their thirty-five coefficients, indexed by increasing four-tuples with repetition. Apply all four-index permutations and all coefficient sign vectors with product +1, identifying a sign vector with its negative. There are 96 actions. The lexicographically least transformed coefficient vector is the representative; sorted representatives give the 84 data indices. The checker enumerates all 2744 triples and checks every source index.

Order quadratic monomials as 00, 01, 02, 03, 11, 12, 13, 22, 23, 33, and upper-triangular Gram entries lexicographically. Coefficient matching in $z^t G z = p$ gives a rational system $A g = b$. At every two-equal-coefficient point where p vanishes, append $G z = 0$. These are constraints for finding a candidate Gram; positivity is subsequently checked, not assumed. Reduced row echelon form specifies pivot and free columns. The data provide integer numerators and a positive denominator for the free entries. Reconstruct the remaining entries and recheck the entire unreduced system.

For a symmetric rational matrix, eliminate a positive pivot d by replacing the remaining block with its Schur complement $M_{22} - M_{21} M_{12} / d$. A zero pivot is accepted only with an entirely zero remaining row. Negative pivots, or nonzero rows at zero pivots, cause failure. The number of positive pivots is the rank. The original curvature certificate has 83 rank-ten matrices and one rank-nine matrix, with kernel spanned, in the displayed monomial order, by $(1, 1, 0, 0, 1, 0, 0, 0, 0, 0)$. Positive semidefiniteness makes $z^t G z = 0$ equivalent to membership in this kernel. If $z = z(x) \neq 0$ is a quadratic monomial vector, its 22 and 33 entries force $x_2 = x_3 = 0$, and its 00, 01, 11 entries force $x_0^2 = x_0 x_1 = x_1^2$, hence $x_0 = x_1 \neq 0$. Undoing the coefficient actions gives the equal-magnitude factorized locus used in the endpoint proof. The modified matrices (5.3) all have rank ten.

A.3 Replay and scope

From the extracted supplement root run

```
python3 verification/run_checks.py
python3 verification/run_checks.py --full
```

The extended run executes the original, margin and moment verifiers in ordinary, -0, and -00 modes. It also deliberately corrupts mixed Gram coefficients, lowers a symbol coefficient below its exact harmonic witness, and changes curvature Gram entries. Each invalid case must fail at its designated mathematical check. Mathematical guards use explicit exceptions and survive Python optimization. The separate reduction checker reconstructs signed averaging, matching tensors, Bianchi identities, and rotated-plane formulas. An independent witness checker reconstructs the full jet and curvature data in [remarks 4.3](#) and [6.3](#), including the 4/45 barrier for variable c at fixed moment weight. It reads no certificate and proves only those necessary bounds, not endpoint sufficiency.

The package records hashes, original Git blob identities, interpreter and SymPy versions, commands, exit statuses, and logs. Hash equality establishes byte identity, not mathematical truth. Exact replays verify the finite parts; conservation, integration, the normal form, covering topology, and the conformal construction are the written arguments. No computational receipt establishes bibliographic priority or unrestricted positive-curvature Hopf positivity.

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